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HYPERBOLICITY FOR CLOSED RELATIONS

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Abstract. Hyperbolicity is a core of dynamics. Shadowness and expansiveness for homeomorphisms have been studied by J. Ombach([3], [4], [5]). We study the hyperbolicity (i.e., expansivity and the shadowing property) and the Anosov relation for a closed relation.

1. Introduction and preliminaries

In this paper, we study whether qualitative properties which were established in flows and homeomorphism dynamics will also be established for compact closed relation and investigate the hyperbolicity and the Anosov relation.

Let $(X_1,\ d_1)$, $(X_2,\ d_2)$ be arbitrary compact metric spaces. A *relation* $f : X_1 \to X_2$ is considered as a map from X_1 to the power set of X_2, that is, each $x \in X_1$ corresponds to a subset $f(x)$ of X_2, or a subset of $X_1 \times X_2$ so that $y \in f(x)$ means $(x,\ y) \in f$. We define the *domain* of f by

$$Dom(f) = \{x \in X_1 \mid f(x) \neq \emptyset\}.$$

For relations $f : X_1 \to X_2$ and $g : X_2 \to X_3$ we define the *inverse* $f^{-1} : X_2 \to X_1$, and the *composition* $g \circ f : X_1 \to X_3$ by

$$x \in f^{-1}(y) \iff y \in f(x),$$

and

$$y \in (g \circ f)(x) \iff z \in f(x) \text{ and } y \in g(z) \text{ for some } z \in X_2.$$

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The usual composition properties of associativity, identity, and inversion generalize to the relation, e.g., $1_{X_2} \circ f = f = f \circ 1_{X_1}$ and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

There are additional algebraic properties as well. For example, composition distributes over the union:

$$(\cup_m g_m) \circ (\cup_n f_n) = \cup_{m,n}(g_m \circ f_n)$$

For $f : X \to X$ we define f^n to be the n-fold composition of f ($n = 0, 1, 2, \cdots$ with $f^0 = 1_X$ and $f^1 = f$ by definition). f^{-n} is defined to be $(f^{-1})^n$ (which equals $(f^n)^{-1}$).

For a relation $f : X_1 \to X_2$ and a subset A of X_1 the *image* $f(A) \subset X_2$ is defined by

$$f(A) = \{y \mid (x,\ y) \in f \text{ for some } x \in A\} = \cup\{f(x) \mid x \in A\}.$$

Definition 1.1. [1] A relation $f : X_1 \to X_2$ is said to be a *closed relation* if it is a closed subset of $X_1 \times X_2$ and $f : X_1 \to X_2$ is said to be a *compact relation* if $f(x)$ is a compact subset of X_2 for any $x \in X_1$.

The identity map $1_X : X \to X$ is identified with the diagonal subset of $X \times X$. The ϵ neighborhoods of the diagonal are important examples of relations which are not functions.

$$V_\epsilon \equiv \{(x_1,\ x_2) \in X \times X \mid d(x_1,\ x_2) < \epsilon\},$$
$$\overline{V}_\epsilon \equiv \{(x_1,\ x_2) \in X \times X \mid d(x_1,\ x_2) \le \epsilon\}$$

V_ϵ is open. $\overline{V}_\epsilon$ is closed although it may be larger than the closure of V_ϵ (i.e. $\overline{V}_\epsilon$ need not equal $cl(V_\epsilon)$).

Theorem 1.2. [1] *Let $f : X_1 \to X_2$ and $g : X_2 \to X_3$ be closed relations.*

(1) *The domain $Dom(f)$ is a closed subset of X_1.*
(2) *The inverse $f^{-1} : X_2 \to X_1$ is a closed relation.*
(3) *The composition $g \circ f : X_1 \to X_3$ is a closed relation.*
(4) *If A is a closed subset of X_1 then the image $f(A)$ is a closed subset of X_2.*
(5) *If B is a closed subset of X_2, then $\{x \mid f(x) \cap B \neq \emptyset\}$ is a closed subset of X_1.*
(6) *If U is an open subset of X_2, then $\{x \mid f(x) \subset U\}$ is an open subset of X_1.*

Corollary 1.3. *Corollary 1.2 Let $f : X_1 \to X_2$ be a closed relation. For every closed subset A of X_1 and every $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$f \circ \overline{V}_\delta(A) = f(\overline{V}_\delta(A)) \subset V_\epsilon(f(A)) = V_\epsilon \circ f(A).$$

Proof. Since $V_\epsilon(f(A)) = \cup_{y\in f(A)} V_\epsilon(y) = \cup_{y\in f(A)} B(y,\ \epsilon)$ is an open set, $\{x \mid f(x) \subset V_\epsilon(f(A))\}$ is open in X_1 by Theorem 1.2(6) and it contains A. Hence, it contains some δ neighborhood of A. $\square$

2. Hyperbolicity and Anosov relation

Shadowness, expansiveness and hyperbolicity for homeomorphisms have been studied by Jerzy Ombach([3], [4], [5]). In this section, we study the hyperbolicity (i.e., expansivity and the shadowing property) and the Anosov relation for a closed relation.

Let $(X,\ d)$ be a compact metric space and f be a closed relation on X whose domain is X. On the product space $X^{\mathbb{Z}}$ we will use the metric, defined by Miller and Akin,

$$\rho(\mathrm{x},\ \mathrm{y}) = \sup\{\min\{d(x_i,\ y_i),\ \frac{1}{|i|}\} \mid i \in \mathbb{Z}\} \tag{2.1}$$

for all x$= (x_i)_{i\in\mathbb{Z}}$, y$= (y_i)_{i\in\mathbb{Z}}$, with $\min\{a,\ \frac{1}{0}\} = a$ by convention.

To show that $X^{\mathbb{Z}}$ is metrizable, we first need the following Proposition 2.1:

Proposition 2.1. *Let* x, y $\in X^{\mathbb{Z}}$ *and* $\epsilon > 0$ *be given. Then* $\rho(\mathrm{x},\ \mathrm{y}) \leq \epsilon$ *if and only if* $d(x_i,\ y_i) \leq \epsilon$ *for all* i *such that* $|i| < \frac{1}{\epsilon}$.

Proof. For given $\epsilon > 0$, suppose that $\rho(\mathrm{x},\ \mathrm{y}) \leq \epsilon$ for some x, y $\in X^{\mathbb{Z}}$. It is trivial that

$$d(x_0,\ y_0) = \min\{d(x_0,\ y_0),\ \frac{1}{0}\} \leq \rho(\mathrm{x},\ \mathrm{y}) \leq \epsilon,$$

for $i = 0$.

Let $0 < |i| < \frac{1}{\epsilon}$, then $\frac{1}{|i|} > \epsilon$. If $d(x_i,\ y_i) \geq \frac{1}{|i|}$, then

$$\epsilon \geq \rho(\mathrm{x},\ \mathrm{y}) \geq \min\{d(x_i,\ y_i),\ \frac{1}{|i|}\} = \frac{1}{|i|} > \epsilon,$$

we have a contradiction. Thus $d(x_i,\ y_i) < \frac{1}{|i|}$. Hence

$$d(x_i,\ y_i) = \min\{d(x_i,\ y_i),\ \frac{1}{|i|}\} \leq \rho(\mathrm{x},\ \mathrm{y}) \leq \epsilon.$$

Suppose that $d(x_i,\ y_i) \leq \epsilon$ for all $|i| < \frac{1}{\epsilon}$, then

$$\min\{d(x_i,\ y_i),\ \frac{1}{|i|}\} \leq d(x_i,\ y_i) \leq \epsilon.$$

Let $|i| \geq \frac{1}{\epsilon}$. Since $\frac{1}{|i|} \leq \epsilon$, we have

$$\min\{d(x_i,\ y_i),\ \frac{1}{|i|}\} \leq \frac{1}{|i|} \leq \epsilon.$$

Thus $\rho(\mathrm{x},\ \mathrm{y}) \leq \epsilon$. □

Proposition 2.2. *ρ is a metric that induces the product topology on $X^{\mathbb{Z}}$.*

Proof. First, we prove that ρ is a metric on $X^{\mathbb{Z}}$. Let $\mathrm{x}, \mathrm{y}, \mathrm{z} \in X^{\mathbb{Z}}$. $\rho(\mathrm{x},\ \mathrm{y}) \geq 0$ is trivial. If $\mathrm{x} = \mathrm{y}$, then $\rho(\mathrm{x},\ \mathrm{y}) = 0$. If $\rho(\mathrm{x},\ \mathrm{y}) = 0$ and $\mathrm{x} \neq \mathrm{y}$, then there exists $i \in \mathbb{Z}$ such that $x_i \neq y_i$. If $i = 0$, we have that

$$\rho(\mathrm{x},\ \mathrm{y}) \geq \min\{d(x_0,\ y_0),\ \frac{1}{0}\} = d(x_0,\ y_0) > 0.$$

This is a contradiction. If $i \neq 0$, we have

$$\rho(\mathrm{x},\ \mathrm{y}) \geq \min\{d(x_i,\ y_i),\ \frac{1}{|i|}\} > 0$$

because $d(x_i,\ y_i) > 0$ and $\frac{1}{|i|} > 0$. This is a contradiction. Therefore $\mathrm{x} = \mathrm{y}$. $\rho(\mathrm{x},\ \mathrm{y}) = \rho(\mathrm{y},\ \mathrm{x})$ is clear. For $i = 0$, we have

$$\begin{aligned} \min\{d(x_0,\ y_0),\ \frac{1}{0}\} &= d(x_0,\ y_0) \leq d(x_0,\ z_0) + d(z_0,\ y_0) \\ &= \min\{d(x_0,\ z_0),\ \frac{1}{0}\} + \min\{d(z_0,\ y_0),\ \frac{1}{0}\} \\ &\leq \rho(\mathrm{x},\ \mathrm{z}) + \rho(\mathrm{z},\ \mathrm{y}) \end{aligned}$$

For $i \neq 0$, we have

$$\begin{aligned} \min\{d(x_i,\ y_i),\ \frac{1}{|i|}\} &\leq d(x_i,\ y_i) \leq d(x_i,\ z_i) + d(z_i,\ y_i) \\ &= \min\{d(x_i,\ z_i),\ \frac{1}{|i|}\} + \min\{d(z_i,\ y_i),\ \frac{1}{|i|}\} \\ &\leq \rho(\mathrm{x},\ \mathrm{z}) + \rho(\mathrm{z},\ \mathrm{y}) \end{aligned}$$

when $d(x_i,\ z_i) \leq \frac{1}{|i|}$, $d(z_i,\ y_i) \leq \frac{1}{|i|}$, and

$$\begin{aligned} \min\{d(x_i,\ y_i),\ \frac{1}{|i|}\} &\leq \frac{1}{|i|} \\ &= \min\{d(x_i,\ z_i),\ \frac{1}{|i|}\} \text{ or } \min\{d(z_i,\ y_i),\ \frac{1}{|i|}\} \\ &\leq \rho(\mathrm{x},\ \mathrm{z}) \text{ or } \rho(\mathrm{z},\ \mathrm{y}) \\ &\leq \rho(\mathrm{x},\ \mathrm{z}) + \rho(\mathrm{z},\ \mathrm{y}) \end{aligned}$$

when $d(x_i,\ z_i) \geq \frac{1}{|i|}$ or $d(z_i,\ y_i) \geq \frac{1}{|i|}$.

Hence $\rho(\mathrm{x},\ \mathrm{y}) \leq \rho(\mathrm{x},\ \mathrm{z}) + \rho(\mathrm{z},\ \mathrm{y})$ for all x, y, z $\in X^{\mathbb{Z}}$. Therefore ρ is a metric on $X^{\mathbb{Z}}$.

Let $\Im_\rho$ be the topology induced by ρ and let $\Im_p$ be the product topology on $X^{\mathbb{Z}}$.

To show that $\Im_\rho = \Im_p$, let $U \in \Im_\rho$. For all x $\in U$, there exists an $\epsilon > 0$ such that $B_\rho(\mathrm{x},\ \epsilon) \subset U$. Since $\epsilon > 0$, we can choose a natural number n such that $\frac{2}{\epsilon} < n$. Let $V_i \equiv B_d(x_i,\ \frac{\epsilon}{2})$ for $-n \leq i \leq n$, and $V_i = X$ for $|i| > n$. Then $V \equiv \prod_{i=-\infty}^{\infty} V_i$ is a basic neighborhood of x in $\Im_p$. If y $\in V$, since $d(x_i,\ y_i) < \frac{\epsilon}{2}$ for all i such that $|i| < \frac{2}{\epsilon}$, we obtain $\rho(\mathrm{x},\ \mathrm{y}) \leq \frac{\epsilon}{2} < \epsilon$ by Proposition 2.1. Thus $V \subset B_\rho(\mathrm{x},\ \epsilon) \subset U$. This means $\Im_\rho \subset \Im_p$.

Let $U \in \Im_p$ and x $\in U$. By definition of the product topology, there exists basic open set $V = \prod_{i=-\infty}^{\infty} V_i$ in $\Im_p$ such that $x \in V \subset U$. We can find natural number n which $V_i = X$ for all $|i| > n$. There exists an $\epsilon > 0$ such that $B_d(x_i,\ \epsilon) \subset V_i$ for all $|i| \leq n$. Choose $\delta > 0$ with $n < \frac{1}{\delta}$ and $\delta < \epsilon$. If $\rho(\mathrm{x},\ \mathrm{y}) < \delta$, then $d(x_i,\ y_i) < \delta < \epsilon$ for all i such that $|i| < \frac{1}{\delta}$ by Proposition 2.1. This means $y_i \in B_d(x_i,\ \epsilon) \in V_i$ for all $|i| \leq n$. Thus y $\in \prod_{i=-\infty}^{\infty} V_i = V \subset U$. i.e., $B_\rho(\mathrm{x},\ \delta) \subset U$. □

We denote by σ the *shift homeomorphism* on $X^{\mathbb{Z}}$ and by $\pi_0 : X^{\mathbb{Z}} \to X$ the projection on the 0-th coordinate.

PROPOSITION 2.3. *Let* x, y $\in X^{\mathbb{Z}}$. *Then*

$$\sup\{d(x_i,\ y_i) \mid i \in \mathbb{Z}\} = \sup\{\rho(\sigma^i(\mathrm{x}),\ \sigma^i(\mathrm{y})) \mid i \in \mathbb{Z}\}.$$

Proof. Let x, y $\in X^{\mathbb{Z}}$. If x=y, then $\sup\{d(x_i,\ y_i) \mid i \in \mathbb{Z}\} = \sup\{\rho(\sigma^i(\mathrm{x}),\ \sigma^i(\mathrm{y})) \mid i \in \mathbb{Z}\} = 0$.

Suppose that there is a $p > 0$ such that

$$\sup\{d(x_i,\ y_i) \mid i \in \mathbb{Z}\} < p < \sup\{\rho(\sigma^i(\mathrm{x}),\ \sigma^i(\mathrm{y})) \mid i \in \mathbb{Z}\}.$$

Then there exists a $j \in \mathbb{Z}$ such that

$$p < \rho(\sigma^j(\mathrm{x}),\ \sigma^j(\mathrm{y})) = \sup\{\min\{d(\sigma^j(\mathrm{x})_i,\ \sigma^j(\mathrm{y})_i),\ \frac{1}{|i|}\} \mid i \in \mathbb{Z}\}.$$

By the definition of the sup, there exists a $k \in \mathbb{Z}$ such that

$$p < \min\{d(\sigma^j(\mathrm{x})_k,\ \sigma^j(\mathrm{y})_k),\ \frac{1}{|k|}\}.$$

By the way,

$$\begin{aligned} p > \sup\{d(x_i,\ y_i) \mid i \in \mathbb{Z}\} &\geq d(x_{j+k},\ y_{j+k}) \\ &= d(\sigma^j(\mathrm{x})_k,\ \sigma^j(\mathrm{y})_k) \\ &\geq \min\{d(\sigma^j(\mathrm{x})_k,\ \sigma^j(\mathrm{y})_k),\ \frac{1}{|k|}\} \\ &> p. \end{aligned}$$

This is a contradiction. Thus

$$\sup\{d(x_i,\ y_i) \mid i \in \mathbb{Z}\} \geq \sup\{\rho(\sigma^i(\mathrm{x}),\ \sigma^i(\mathrm{y})) \mid i \in \mathbb{Z}\}.$$

Suppose that there is a $q > 0$ such that

$$\sup\{d(x_i,\ y_i) \mid i \in \mathbb{Z}\} > q > \sup\{\rho(\sigma^i(\mathrm{x}),\ \sigma^i(\mathrm{y})) \mid i \in \mathbb{Z}\}.$$

Then there exists a $j \in \mathbb{Z}$ such that $d(x_j,\ y_j) > q$.

$$\begin{aligned} q &> \sup\{\rho(\sigma^i(\mathrm{x}),\ \sigma^i(\mathrm{y})) \mid i \in \mathbb{Z}\} \\ &\geq \rho(\sigma^j(\mathrm{x}),\ \sigma^j(\mathrm{y})) \\ &\geq \min\{d(\sigma^j(\mathrm{x})_0,\ \sigma^j(\mathrm{y})_0),\ \frac{1}{0}\} \\ &= d(\sigma^j(\mathrm{x})_0,\ \sigma^j(\mathrm{y})_0) \\ &= d(x_j,\ y_j) > q. \end{aligned}$$

This is a contradiction. Thus

$$\sup\{d(x_i,\ y_i) \mid i \in \mathbb{Z}\} \leq \sup\{\rho(\sigma^i(\mathrm{x}),\ \sigma^i(\mathrm{y})) \mid i \in \mathbb{Z}\}.$$

□

The *sample path space* for f is the subspace X_f of $X^{\mathbb{Z}}$ defined by the condition

$$\mathrm{x} \in X_f \iff (x_i,\ x_{i+1}) \in f$$

for all $i \in \mathbb{Z}$.

Proposition 2.4. *X_f is a closed invariant subset of $X^{\mathbb{Z}}$.*

Proof. Let $\mathrm{x} \in \overline{X_f}$. Then there exists a sequence (x^n) in X_f such that $\mathrm{x}^n \to \mathrm{x}$. For each $i \in \mathbb{Z}$, since $(x_i^n,\ x_{i+1}^n) \in f$ and $(x_i^n,\ x_{i+1}^n) \to (x_i,\ x_{i+1})$, we have $(x_i,\ x_{i+1}) \in \overline{f} = f$. Thus $\mathrm{x} \in X_f$. Hence X_f is closed in $X^{\mathbb{Z}}$. It is clear that X_f is invariant. □

The homeomorphism σ_f on X_f is obtained by restricting the corresponding shift. The restriction of the projection is denoted by $\pi_0 : X_f \to X$.

A relation f on X is called *surjective* if $f(X) = X$.

For a closed subset A of X the restriction of f to $A \times A$ is

$$f_A = f \cap (A \times A).$$

The sample path space of f_A is $A_f = X_f \cap A^{\mathbb{Z}}$.

Theorem 2.5. *Let f be a closed relation on X. Then*

$$\pi_0(X_f) = \cap_{i=-\infty}^{\infty} f^i(X). \tag{2.2}$$

Proof. Let x $\in X_f$. Since $\pi_0(\mathrm{x}) = x_0 \in f^i(x_{-i}) \subset f^i(X)$ for all $i \in \mathbb{Z}$, we have

$$\pi_0(X_f) \subset \cap_{i=-\infty}^{\infty} f^i(X).$$

Let $x \in \cap_{i=-\infty}^{\infty} f^i(X)$. For each positive integer n, there exist $x_n,\ x_{-n} \in X$ such that

$$x \in f^n(x_{-n}) \cap f^{-n}(x_n).$$

Define $x_{-n}^n = x_{-n},\ x_0^n = x,\ x_n^n = x_n$. Since $x_0^n \in f^n(x_{-n}^n)$ and $x_n^n \in f^n(x_0^n)$, there exist

$$x_{-n+1}^n, \cdots, x_{-1}^n, x_1^n, \cdots,\ x_{n-1}^n \in X$$

such that $x_{i+1}^n \in f(x_i^n)$ for $-n \le i < n$. For each $i \in \mathbb{Z}$, the sequence $(x_i^n)_{n \ge |i|}$ has a convergent subsequence. Let $x_i^n \to x_i$ as $n \to \infty$. Since

$(x_i^n,\ x_{i+1}^n) \in f$ and $(x_i^n,\ x_{i+1}^n) \to (x_i,\ x_{i+1})$ as $n \to \infty$,

we have $(x_i,\ x_{i+1}) \in \overline{f} = f$. Thus x$=(x_i)_{i\in\mathbb{Z}} \in X_f$ and $x = x_0 \in \pi_0(X_f)$.

This proves Theorem 2.5. □

This set $\pi_0(X_f) = \cap_{i=-\infty}^{\infty} f^i(X)$, denoted by $D(f)$, is called the *dynamic domain* of f.

Proposition 2.6. *For a closed subset A of X the following conditions are equivalent and when they hold A is called a* surjective subset *of X.*

(1) *f_A is a surjective relation on A.*
(2) *$A \subset f(A) \cap f^{-1}(A)$.*
(3) *$\pi_0(A_f) = A$.*
(4) *There exists a σ_f-invariant subset K of X_f such that $\pi_0(K) = A$.*

The dynamic domain of f is the maximum surjective subset of X, that is, if A is a surjective subset of X then $A \subset D(f)$. In particular, f is surjective if and only if $D(f) = X$.

Proof. Clearly, if f is surjective then $\pi_0(X_f) = X$. In particular, applied to f_A we get (1) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (4) is obvious. To prove (4) $\Rightarrow$ (2) let $x \in A$ and choose x$\in K$ such that $x_0 = x$. Since x$\in X_f$, we have $x = x_0 \in f(x_{-1}) \cap f^{-1}(x_1)$. By the invariance of K,

$\sigma_f^{-1}(\mathrm{x})$, $\sigma_f(\mathrm{x}) \in K$ and so $x_{-1} = \pi_0(\sigma_f^{-1}(\mathrm{x}))$ and $x_1 = \pi_0(\sigma_f(\mathrm{x}))$ are in $\pi_0(K) = A$. Thus $x \in f(A) \cap f^{-1}(A)$. If $A \subset f(A) \cap f^{-1}(A) \subset f(A)$, then

$$A = f(A) \cap A = f_A(A).$$

This proves (2) $\Rightarrow$ (1). □

REMARK. In general, if K is an σ_f-invariant subset of X_f such that $\pi_0(K) \subset A$ then $K \subset A_f$. That is, A_f is the maximum σ_f-invariant subset of $\pi_0^{-1}(A)$ in X_f.

For any closed subset A of X

(2.3) $$D(f_A) = \pi_0(A_f) = \cap_{i=-\infty}^{\infty} f_A^i(A).$$

is the maximum surjective subset of A.

LEMMA 2.7. *For closed subsets A and B of X the following conditions are equivalent :*

(1) $D(f_A) \subset D(f_B)$
(2) $D(f_A) \subset B$
(3) $A_f \subset \pi_0^{-1}(B)$
(4) $A_f \subset B_f$

Proof. Since $D(f_B) \subset B$, (1)$\Rightarrow$(2) is clear. Since $\pi_0(A_f) \subset B$ if and only if $A_f \subset \pi_0^{-1}(B)$, (2)$\Rightarrow$(3) is obvious. B_f is the maximum σ_f-invariant subset of $\pi_0^{-1}(B)$. Since A_f is σ_f-invariant, (3) implies (4). By definition of the dynamic domain of f, (4) implies (1). □

Let f and g be closed relations on X and Y, respectively. A continuous map $h : X \to Y$ is said to *map f to g*, written $h : f \to g$ if $(x_1,\ x_2) \in f$ implies $(h(x_1),\ h(x_2)) \in g$. This condition is equivalent to the following inclusion:

$$h \circ f \subset g \circ h$$

A continuous map $h : X \to Y$ is called a *semiconjugacy* from f to g if h is onto and $h \circ f = g \circ h$. A *conjugacy* is a homeomorphism $h : X \to Y$ such that h maps f to g and h^{-1} maps g to f, or equivalently a homeomorphism h such that

$$h \circ f = g \circ h.$$

If h maps f to g, then the induced map $h_* : X^{\mathbb{Z}} \to Y^{\mathbb{Z}}$ defined by $h_*(\mathrm{x})_i = h(x_i)$ satisfies $h_*(X_f) \subset Y_g$.

THEOREM 2.8. *Let f and g be closed relations on X and Y, respectively ; let a continuous map $h : X \to Y$ map f to g ; and let A and B be closed subsets of X and Y respectively.*

(1) *If A is surjective with respect to f, then $B = h(A)$ is surjective with respect to g.*
(2) *If h is a conjugacy from f to g, then $h_*(X_f) = Y_g$.*
(3) *If B is surjective with respect to g, $A = h^{-1}(B)$ and h is a semi-conjugacy, then $h(D(f_A)) = B$.*

Proof. (1) For any $y \in B = h(A)$ there exists $x \in A$ such that $y = h(x)$. Since A is surjective, there exist $x_{-1},\ x_1 \in A$ such that $(x_{-1},\ x),\ (x,\ x_1) \in f$. We have

$$h(x_{-1}),\ h(x_1) \in h(A) = B$$
$$(h(x_{-1}),\ h(x_1)) = (h(x_{-1}),\ y),\ (h(x_1),\ h(x)) = (h(x_1),\ y) \in g.$$

Thus B is surjective.

(2) It is clear $h_*(X_f) \subset Y_g$. Let y$\in Y_g$. Since h is onto, there exists $x_i \in X$ such that $h(x_i) = y_i$. Since y$\in Y_g$, $y_{i+1} \in g(y_i) = g(h(x_i)) = (h \circ f)(x_i)$ and there exists $x_{i+1} \in f(x_i)$ such that $h(x_{i+1}) = y_{i+1}$. If y$\in Y_g$ and $n \in \mathbb{Z}_+$, then we can start at y_{-n} and proceed inductively forward to define $x_i^n \in X$ so that $h(x_i^n) = y_i$ and $(x_i^n,\ x_{i+1}^n) \in f$ for all $i \geq -n$. For each $i \in \mathbb{Z}$, the sequence $(x_i^n)_{n \geq |i|}$ has a convergent subsequence. Let $x_i^n \to x_i$ as $n \to \infty$. Then x$= (x_i) \in X_f$ and h_*(x) =y. Thus $Y_f \subset h_*(X_f)$. Hence $h_*(X_f) = Y_g$.

(3) From (2) with $A = h^{-1}(B)$ it follows that $h_*(A_f) = B_g$. Now apply $\pi_0 : X_f \to X$. Because $\pi_0 \circ h_* = h \circ \pi_0$ and B is surjective,

$$h(D(f_A)) = h(\pi_0(A_f)) = \pi_0(h_*(A_f)) = \pi_0(B_g) = B.$$

□

A closed subset A of X is called *isolated* (rel a closed subset B of X) with respect to f if there exists a $\gamma > 0$ such that

(2.4) $x \in X_f$and $d(x_i,\ A) \leq \gamma$ for all $i \in \mathbb{Z}$ implies $x_i \in B$ for all $i \in \mathbb{Z}$.

We call A isolated if A is isolated (rel A).

Theorem 2.9. *Let f be a closed relation on X and A, B closed subsets of X.*

(a) *A is isolated (rel B) with respect to f if and only if there exists a closed neighborhood U of A such that the following equivalent conditions hold:*
  (1) $D(f_U) \subset D(f_B)$
  (2) $D(f_U) \subset B$
  (3) $U_f \subset \pi_0^{-1}(B)$
  (4) $U_f \subset B_f$
(b) *The following conditions are equivalent :*
  (1) *A is isolated (rel B) with respect to f.*

(2) *A is isolated (rel $D(f_B)$) with respect to f.*
(3) *$D(f_A)$ is isolated (rel $D(f_B)$) with respect to f.*
(4) *$\pi_0^{-1}(A)$ is isolated (rel $\pi_0^{-1}(B)$) with respect to σ_f.*
(5) *A_f is isolated (rel B_f) with respect to σ_f.*

(c) *Assume g is a closed relation on Y and a continuous map $h : Y \to X$ maps g to f. Let $A_1 = h^{-1}(A)$ and $B_1 = h^{-1}(B)$. If A is isolated (rel B) with respect to f then A_1 is isolated (rel B_1) with respect to g. Conversely, if A_1 is isolated (rel B_1) with respect to g and h is a semiconjugacy then A is isolated (rel B) with respect to f.*

Proof. (a) The equivalences are clear from Lemma 2.7. Condition (2.4) is ture if and only if (4) holds with $U = \{x \in X \mid d(x,\ A) \leq \gamma\}$.

(b) (1) $\Leftrightarrow$ (2) This follows from the equivalence of (1) with (2) in (a).

(2) $\Rightarrow$ (3) If A is isolated (rel B) then any closed subset of A is isolated (rel B).

(3) $\Rightarrow$ (1) Since $D(f_B) \subset B$, $D(f_A)$ is isolated (rel B). By (a), there exists a closed neighborhood G of $D(f_A)$ such that $G_f \subset B_f$. (2.3) and compactness imply that

$$\cap_{k=-N}^{N} f_A^k(A) \subset \mathrm{Int}(G)$$

for some natural number N. Let $U_n = \{x \in X \mid d(x,\ A) \leq \frac{1}{n}\}$. Then (U_n) is a decreasing sequence of closed neighborhood of A with intersection A. Since the sequence (f_{U_n}) of closed relations decreases to f_A, we can find a closed neighborhood $U = U_m$ of A such that

$$\cap_{k=-N}^{N} f_U^k(U) \subset \mathrm{Int}(G). \tag{2.5}$$

Let x$\in U_f$. By (2.5) we have $x_i \in G$ for all $i \in \mathbb{Z}$. Thus x$\in G_f \subset B_f$. Hence we have $U_f \subset B_f$ and so by (a) A is isolated (rel B).

Before completing the proof of (b) we prove (c).

If A is isolated (rel B), then $U_f \subset B_f$ for some closed neighborhood U of A. Let $U_1 = h^{-1}(U)$. Then U_1 is a closed neighborhood of $A_1 = h^{-1}(A)$. If x$\in (U_1)_g$, then $h_*(\mathrm{x}) \in h_*(Y_g) = X_f$. Since $h_*(\mathrm{x})_i = h(x_i) \in h(U_1) = h(h^{-1}(U)) \subset U$ for all $i \in \mathbb{Z}$, we have $h_*(\mathrm{x}) \in U_f \subset B_f$. Thus $h(x_i) = h_*(\mathrm{x})_i \in B$ implying $x_i \in h^{-1}(B) = B_1$ for all $i \in \mathbb{Z}$. Hence x$\in (B_1)_g$ so $(U_1)_g \subset (B_1)_g$. Therefore A_1 is isolated (rel B_1).

Assume A_1 is isolated (rel B_1). Then $(U_1)_g \subset (B_1)_g$ for some closed neighborhood U_1 of $A_1 = h^{-1}(A)$. By compactness, there exists a closed neighborhood U of A such that $h^{-1}(U) \subset U_1$. Let x$\in U_f$. Since $h_*(Y_g) = X_f$, there exists y$\in Y_g$ such that $h_*(\mathrm{y})$=x. We have $h(y_i) = h_*(\mathrm{y})_i = x_i \in U$ and so $y_i \in h^{-1}(U) \subset U_1$ for all $i \in \mathbb{Z}$. Thus y$\in (U_1)_g \subset (B_1)_g$.

Hence $y_i \in B_1 = h^{-1}(B)$ and $h(y_i) = x_i \in B$ for all $i \in \mathbb{Z}$, that is, x$\in B_f$. Therefore $U_f \subset B_f$ and so A is isolated (rel B).

Returning to (b), (1) $\Leftrightarrow$ (4) The continuous map $\pi_0 : X_f \to X$ maps σ_f to f. Since $\pi_{0*}((X_f)_{\sigma_f}) = X_f$, the equivalence of (1) with (4) follows from (c).

(4) $\Leftrightarrow$ (5) A_f is the maximum σ_f-invariant subset of $\pi_0^{-1}(A)$ and similarly for B_f. Thus the equivalence of (4) with (5) is just (1) $\Leftrightarrow$ (3) applied to σ_f. □

$f \times f$ is a closed relation on $X \times X$ defined by

$$f \times f(x_1,\ x_2) = (f(x_1),\ f(x_2)).$$

If A is a closed subset of X then A is surjective with respect to f if and only if $A \times A$ is surjective with respect to $f \times f$ if and only if 1_A is surjective with respect to $f \times f$.

A closed subset A of X is called *expansive* for f if 1_A is isolated (rel 1_X) with respect to $f \times f$. That is, there exists a $\gamma > 0$ (called *expansive constant* for A) such that

(2.6)

$$\text{x, y} \in X_f \text{ and} \max(d(x_i,\ A),\ d(y_i,\ A),\ d(x_i,\ y_i)) \leq \gamma \text{ for all } i \in \mathbb{Z}$$

implies $x = y$.

f is called an expansive relation if X is expansive, that is, 1_X is isolated with respect to $f \times f$.

Theorem 2.10. *Let $h : X \to Y$ be a semiconjugacy from a closed relation f on X to the closed relation g on Y. Then g is an expansive relation if and only if $h^{-1} \circ h$ is an isolated subset of $X \times X$.*

Proof. We will prove that $h^{-1} \circ h = (h \times h)^{-1}(1_Y)$. Let $(x,\ y) \in h^{-1} \circ h$. Then there exists $z \in Y$ such that $(x,\ z) \in h$ and $(z,\ y) \in h^{-1}$. Then $(x,\ z),\ (y,\ z) \in h$ and so $h(x) = z = h(y)$. Since $(h \times h)(x,\ y) = (h(x),\ h(y)) = (z,\ z)$, we have

$$(x,\ y) = (h \times h)^{-1}(z,\ z) \in (h \times h)^{-1}(1_Y).$$

Let $(x,\ y) \in (h \times h)^{-1}(1_Y)$. Then there exists $(z,\ z) \in 1_Y$ such that

$$(x,\ y) = (h \times h)^{-1}(z,\ z).$$

Since $(z,\ z) = h \times h(x,\ y) = (h(x),\ h(y))$, we have $(x,\ z),\ (y,\ z) \in h$. Then

$$(x,\ z) \in h \text{ and } (z,\ y) \in h^{-1}.$$

Thus $(x,\ y) \in h^{-1} \circ h$.

g is expansive if and only if 1_Y is isolated with respect to $g \times g$. Since $h \times h$ is a semiconjugacy from $f \times f$ to $g \times g$, by Theorem 2.9(c), 1_Y is isolated for $g \times g$ if and only if $(h \times h)^{-1}(1_Y) = h^{-1} \circ h$ is isolated for $(h \times h)^{-1}(g \times g) = f \times f$. $\square$

Let $\gamma \geq 0$. An element x of $X^{\mathbb{Z}}$ is called a γ-*chain* for f if

$$d(x_{i+1},\ f(x_i)) \leq \gamma \text{ for all } i \in \mathbb{Z}.$$

An element x of $X^{\mathbb{Z}}$ is said to γ-*shadow* an element y of $X^{\mathbb{Z}}$ if

$$d(x_i,\ y_i) \leq \gamma \text{ for all } i \in \mathbb{Z}.$$

If A is a surjective closed subset of X then A satisfies the *shadowing property* in X if for every $\epsilon > 0$ there exists a $\delta > 0$ such that any δ-chain for f in A is ϵ-shadowed by some 0-chain in X. That is if x$\in A^{\mathbb{Z}}$ with $d(x_{i+1},\ f(x_i)) \leq \delta$ for all $i \in \mathbb{Z}$, then there exists y$\in X_f$ such that $d(x_i,\ y_i) \leq \epsilon$ for all $i \in \mathbb{Z}$.

We will need a pair of technical lemmas.

LEMMA 2.11. *Let A be a closed subset of X. For every $\epsilon > 0$ there exists a $\delta > 0$ such that every δ-chain for f in $\overline{V}_\delta(A)$ is $\frac{\epsilon}{2}$-shadowed by some ϵ-chain for f_A.*

Proof. In $A \times A$, $\overline{V}_{\frac{\epsilon}{2}} \circ f_A \circ \overline{V}_{\frac{\epsilon}{2}}$ is a neighborhood of the compact set f_A. Since

$$(\overline{V}_\delta \circ f) \cap (\overline{V}_\delta(A) \times \overline{V}_\delta(A)) \to f_A \text{ as } \delta \to 0,$$

there exists a $\delta > 0$ such that

$$(\overline{V}_\delta \circ f) \cap (\overline{V}_\delta(A) \times \overline{V}_\delta(A)) \subset \overline{V}_{\frac{\epsilon}{2}} \circ f_A \circ \overline{V}_{\frac{\epsilon}{2}}.$$

If x$\in \overline{V}_\delta(A)^{\mathbb{Z}}$ is a δ-chain, then

$$(x_i,\ x_{i+1}) \in (\overline{V}_\delta \circ f) \cap (\overline{V}_\delta(A) \times \overline{V}_\delta(A)) \text{ for all } i \in \mathbb{Z}.$$

and so there exists $y_i \in A$ such that

$$d(x_i,\ y_i) \leq \frac{\epsilon}{2} \text{ and } d(x_{i+1},\ f_A(y_i)) \leq \frac{\epsilon}{2} \text{ for all } i \in \mathbb{Z}.$$

Thus

$$d(y_{i+1},\ f(y_i)) \leq d(y_{i+1},\ x_{i+1}) + d(x_{i+1},\ f(y_i)) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence y$= (y_i) \in A^{\mathbb{Z}}$ is an ϵ-chain for f_A and $\frac{\epsilon}{2}$-shadows x. $\square$

Corollary 2.12. *Let f be a closed relation on X and A be a surjective subset of X. A satisfies the shadowing property in X if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that any δ-chain for f_A is ϵ-shadowed by some 0-chain for f in X. That is, if* $\mathrm{x} \in A^{\mathbb{Z}}$ *with* $d(x_{i+1},\ f(x_i) \cap A) \le \delta$ *for all* $i \in \mathbb{Z}$, *then there exists* $\mathrm{y} \in X_f$ *such that* $d(x_i,\ y_i) \le \epsilon$ *for all* $i \in \mathbb{Z}$.

Proof. Assume δ_1-chains for f_A are $\frac{\epsilon}{2}$-shadowed by 0-chains for f. Use Lemma 2.11 with ϵ replaced by $\min\{\frac{\epsilon}{2},\ \delta_1\}$ choose $\delta > 0$ so that any δ-chain for f in A can be $\frac{\epsilon}{2}$-shadowed by a δ_1-chain for f_A. Thus any δ-chain for f in A is ϵ-shadowed by some 0-chain for f.

The converse is obvious. □

Let f be a relation on X. f is said to be *upper semicontinuous* if for any $x \in X$ and any $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $f(y) \subset B_d(f(x), \epsilon)$. f is said to be *lower semicontinuous* if for any $x \in X$ and any $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $f(x) \subset B_d(f(y), \epsilon)$. f is said to be *continuous* if f is upper and lower semicontinuous.

Proposition 2.13. *A closed relation f on X is upper semicontinuous.*

Proof. Assume that f is not upper semicontinuous. Then there exist $x \in X$ and $\epsilon > 0$ such that for any $\delta > 0$ there exists $y \in B_d(x, \delta)$ such that $f(y) \not\subset B_d(f(x), \epsilon)$. For each n, there exists $x_n \in B_d(x, \frac{1}{n})$ such that $f(x_n) \not\subset B_d(f(x), \epsilon)$. We can choose $y_n \in f(x) - B_d(f(x), \epsilon)$. Since X is compact, the sequence (y_n) has a convergent subsequence. Let $y_n \to y$ as $n \to \infty$. Since $(x_n, y_n) \in f$ and $(x_n, y_n) \to (x, y)$ as $n \to \infty$, we have $(x, y) \in \overline{f} = f$ that is $y \in f(x)$. Since $d(y_n, f(x)) \ge \epsilon$ for all n, we have $d(y, f(x)) \ge \epsilon$. This is a contradiction. Thus f is upper semicontinuous. □

In the remainder of this paper, we assume that relations are lower semicontinuous.

Proposition 2.14. *Let f be a lower semicontinuous closed surjective relation on X. Given any integer $n \ge 2$ and any $\epsilon > 0$ there exists $\delta > 0$ such that if $(y_1, \cdots, y_n)$ is a δ-chain for f then there exists* $\mathrm{x} \in X_f$ *such that* $d(y_i, x_i) < \epsilon$ *for all* $i = 1, \cdots, n$.

Proof. Step 1. We will prove that for any $\epsilon > 0$ there exists $\eta > 0$ such that if $d(x, y) < \eta$ then $f(x) \subset B_d(f(y), \epsilon)$ and $f(y) \subset B_d(f(x), \epsilon)$.

Let $\epsilon > 0$. For each $x \in X$ there exists $\eta_x > 0$ such that if $d(x,y) < \eta_x$ then

$$f(x) \subset B_d(f(y), \frac{\epsilon}{2}) \text{ and } f(y) \subset B_d(f(x), \frac{\epsilon}{2}).$$

$\{B_d(x, \frac{\eta_x}{2}) | x \in X\}$ is an open cover of X. Since X is compact, there exist $x_1, \cdots, x_n \in X$ such that $X = \bigcup_{i=1}^n B_d(x_i, \frac{\eta_i}{2})$ where $\eta_i = \eta_{x_i}$. Put

$$\eta = \min\{\frac{\eta_1}{2}, \cdots, \frac{\eta_n}{2}\}.$$

Let $x \in X$ and $d(x,y) < \eta$. There exists i such that $x \in B_d(x_i, \frac{\eta_i}{2})$. Since $d(x_i, x) < \frac{\eta_i}{2} < \eta_i$, we have $f(x_i) \subset B_d(f(x), \frac{\epsilon}{2})$ and $f(x) \subset B_d(f(x_i), \frac{\epsilon}{2})$. Since

$$d(x_i, y) \leq d(x_i, x) + d(x, y) < \frac{\eta_i}{2} + \eta \leq \frac{\eta_i}{2} + \frac{\eta_i}{2} = \eta_i,$$

we have $f(x_i) \subset B_d(f(y), \frac{\epsilon}{2})$ and $f(y) \subset B_d(f(x_i), \frac{\epsilon}{2})$. Thus we have $f(x) \subset B_d(f(x_i), \frac{\epsilon}{2}) \subset B_d(f(y), \epsilon)$ and $f(y) \subset B_d(f(x_i), \frac{\epsilon}{2}) \subset B_d(f(y), \epsilon)$.

Step 2. We prove by induction on n. Assume that Proposition 2.14 holds for n. Given any $\epsilon > 0$, by Step 1, there exists $0 < \eta < \epsilon$ such that if $d(x,y) < \eta$ then

$$f(x) \subset B_d(f(y), \frac{\epsilon}{2}) \text{ and } f(y) \subset B_d(f(x), \frac{\epsilon}{2}).$$

By induction hypothesis, there exists $\gamma > 0$ such that if $(y_1, \cdots, y_n)$ is a γ-chain for f then there exists a z $\in X_f$ such that $d(y_i, z_i) < \eta$ for all $i = 1, \cdots, n$. Put

$$\delta = \min\{\gamma, \frac{\epsilon}{2}\}.$$

Let $(y_1, \cdots, y_{n+1})$ be a δ-chain for f. Since $(y_1, \cdots, y_n)$ is a γ-chain for f, there exists a z $\in X_f$ such that $d(y_i, z_i) < \eta$ for all $i = 1, \cdots, n$. Since $d(y_n, z_n) < \eta$, we have $f(y_n) \subset B_d(f(z_n), \frac{\epsilon}{2})$ and $f(z_n) \subset B_d(f(y_n), \frac{\epsilon}{2})$. Since $d(y_{n+1}, f(y_n)) < \delta \leq \frac{\epsilon}{2}$, there exists $p \in f(y_n)$ such that $d(y_{n+1}, p) < \frac{\epsilon}{2}$. Since $p \in f(y_n) \subset B_d(f(z_n), \frac{\epsilon}{2})$, there exists $q \in f(z_n)$ such that $d(p,q) < \frac{\epsilon}{2}$. We have

$$d(y_{n+1}, q) \leq d(y_{n+1}, p) + d(p, q) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Define $x_i = z_i$ for $i \leq n$, $x_{n+1} = q$, $x_{i+1} \in f(x_i)$ for $i \geq n+1$. Then x $= (x_i)_{i \in \mathbb{Z}} \in X_f$ and

$$d(y_i, x_i) < \epsilon \text{ for all } i = 1, \cdots, n+1.$$

This completes the proof of Proposition 2.14. □

LEMMA 2.15. *Let* $0 < \epsilon < 1$.

(a) *Assume* $(\mathrm{x}^i)_{i\in\mathbb{Z}}$ *is an* ϵ*-chain for* σ_f *that is* $\mathrm{x}^i \in X_f$ *and* $\rho(\sigma_f(\mathrm{x}^i),\ \mathrm{x}^{i+1}) \le \epsilon$ *for all* $i \in \mathbb{Z}$. *Let* $y_i = \mathrm{x}_0^i = \pi_0(\mathrm{x}^i)$ *for each* $i \in \mathbb{Z}$, *then* $\mathrm{y} = (y_i)_{i\in\mathbb{Z}} \in X^{\mathbb{Z}}$ *is an* ϵ*-chain for* f *and* $\rho(\sigma^i(\mathrm{y}),\ \mathrm{x}^i) \le \sqrt{\epsilon}$ *for all* $i \in \mathbb{Z}$.

(b) *Assume* f *is surjective. There exists a* δ *with* $0 < \delta \le \epsilon$ *such that if* $\mathrm{y} \in X^{\mathbb{Z}}$ *is a* δ*-chain for* f, *then there exists an* ϵ*-chain* $(\mathrm{x}^i)_{i\in\mathbb{Z}}$ *for* σ_f *such that*

$$\rho(\sigma^i(\mathrm{y}),\ \mathrm{x}^i) \le \epsilon \text{ for all } i \in \mathbb{Z}. \tag{2.7}$$

Proof. (a) Since $\mathrm{x}^i \in X_f$, $x_1^i \in f(x_0^i)$ and so

$$d(y_{i+1}, f(y_i)) \le d(x_0^{i+1},\ x_1^i) = d(x_0^{i+1},\ \sigma_f(\mathrm{x}^i)_0) \le \rho(\mathrm{x}^{i+1},\ \sigma_f(\mathrm{x}^i)) \le \epsilon.$$

Thus y is an ϵ-chain for f.

Let $|j| < \frac{1}{\sqrt{\epsilon}}$. We have

$$\begin{aligned} d(\sigma^i(\mathrm{y})_j,\ x_j^i) &= d(y_{i+j},\ x_j^i) \\ &= d(x_0^{i+j},\ x_j^i) \\ &\le \sum_k d(x_{k+1}^{i+j-k-1},\ x_k^{i+j-k}) \\ &= \sum_k d(\sigma_f(\mathrm{x}^{i+j-k-1})_k,\ x_k^{i+j-k}) \end{aligned}$$

where the summation is over $0 \le k < j$ if $j > 0$ and over $j \le k < 0$ if $j < 0$. Since $(\mathrm{x}^i)_{i\in\mathbb{Z}}$ is an ϵ-chain for σ_f, $\rho(\sigma_f(\mathrm{x}^{j+j-k-1}),\ \mathrm{x}^{i+j-k}) \le \epsilon$. Since $|k| \le |j| < \frac{1}{\sqrt{\epsilon}} < \frac{1}{\epsilon}$, $d(\sigma_f(\mathrm{x}^{i+j-k-1})_k,\ x_k^{i+j-k}) \le \epsilon$ by Proposition 2.1. Thus $d(\sigma^i(\mathrm{y})_j,\ x_j^i) \le |j|\epsilon < \sqrt{\epsilon}$. By Proposition 2.1, $\rho(\sigma^i(\mathrm{y}),\ \mathrm{x}^i) \le \sqrt{\epsilon}$ for all $i \in \mathbb{Z}$.

(b) Fix $n > \frac{1}{\epsilon}$. By Proposition 2.14, there exists a $\delta > 0$ such that for every δ-chain y for f and $i \in \mathbb{Z}$ there exists $\mathrm{x}^i \in X_f$ such that

$$d(x_j^i,\ y_{i+j}) \le \frac{\epsilon}{2} \text{ for } |j| \le n. \tag{2.8}$$

In particular for $|j| < \frac{1}{\epsilon}$, we have

$$\begin{aligned} d(\sigma_f(\mathrm{x}^i)_j,\ x_j^{i+1}) &= d(x_{j+1}^i,\ x_j^{i+1}) \\ &\le d(x_{j+1}^i,\ y_{i+j+1}) + d(x_j^{i+1},\ y_{i+j+1}) \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

By Proposition 2.1, $\rho(\sigma_f(\mathrm{x}^i),\ \mathrm{x}^{i+1}) \leq \epsilon$ for all $i \in \mathbb{Z}$ that is (x^i) is an ϵ-chain for σ_f. By (2.8)

$$d(x^i_j,\ \sigma^i(\mathrm{y})_j) \leq \epsilon \text{ for } |j| < \frac{1}{\epsilon}$$

from which (2.7) follows from Proposition 2.1. □

THEOREM 2.16. *Let f be a closed relation on X and A be a surjective subset of X. A satisfies the shadowing property for f if and only if A_f satisfies the shadowing property for σ_f.*

Proof. Assume A satisfies the shadowing property for f. Given any $\epsilon \in (0,\ 1)$, let $\epsilon_1 = (\frac{\epsilon}{2})^2$ and let $\delta \in (0,\ \epsilon_1)$ be such that any δ-chain for f in A is ϵ_1-shadowed by some element of X_f. Let (x^i) be a δ-chain for σ_f in A_f. Define y$\in A^{\mathbb{Z}}$ by $y_i = x^i_0$. By Lemma 2.15(a), y is a δ-chain for f and

$$\rho(\sigma^i(\mathrm{y}),\ \mathrm{x}^i) \leq \frac{\epsilon}{2} \text{ for all } i \in \mathbb{Z}.$$

By the choice of δ, there exists z$\in X_f$ such that $d(y_i,\ z_i) \leq \epsilon_1 < \frac{\epsilon}{2}$ for all $i \in \mathbb{Z}$. Thus we have

$$\rho(\sigma^i(\mathrm{z}),\ \sigma^i(\mathrm{y})) \leq \frac{\epsilon}{2} \text{ for all} i \in \mathbb{Z}.$$

By the triangle inequality, $(\sigma^i(\mathrm{z}))$ is a chain in X_f which ϵ-shadows (x^i).

Assume A_f satisfies the shadowing property for σ_f. Given any $\epsilon \in (0,\ 1]$, let $\epsilon_1 = \frac{\epsilon}{2}$ and choose $\delta_1 \in (0,\ \epsilon_1)$ so that any δ_1-chain for σ_f in A_f can be ϵ_1-shadowed by some 0-chain for σ_f. Since A is a surjective subset of X, the closed relation f_A on A and ϵ replaced by δ_1, choose $\delta \in (0,\ \delta_1)$ satisfies the condition of the Lemma 2.15. Let y be a δ-chain for σ_f. By the choice of δ, there exists a δ_1-chain (x^i) for σ_f such that $\mathrm{x}^i \in A_f$ and $\rho(\sigma^i(\mathrm{y}),\ \mathrm{x}^i) \leq \delta_1$ for all $i \in \mathbb{Z}$. By the choice of δ_1, there exists z$\in X_f$ such that

$$\rho(\sigma^i_f(\mathrm{z}),\ \mathrm{x}^i) \leq \epsilon_1 \text{ for all } i \in \mathbb{Z}.$$

Thus z is a 0-chain for f and

$$\rho(\sigma^i_f(\mathrm{z}),\ \sigma^i(\mathrm{y})) \leq \delta_1 + \epsilon_1 \leq \epsilon \text{ for all } i \in \mathbb{Z}.$$

Hence z ϵ-shadows y. By Corollary 2.12, it follows that A satisfies the shadowing property for f. □

A closed surjective subset A of X is called a *hyperbolic* subset for f if it is an expansive subset which satisfies the shadowing property. This says that there exists a $\gamma > 0$ such that for every ϵ with $0 < \epsilon \leq \gamma$ there exists a $\delta > 0$ so that any δ-chain for f in A is ϵ-shadowed by a unique 0-chain for f in X.

f is called an *Anosov relation* if it is a surjective relation and X is hyperbolic for f.

THEOREM 2.17. *Let f be a closed relation on X and let A be a closed surjective subset of X. The following conditions are equivalent and when they hold we call A an Anosov subset.*

(1) *The restriction f_A is an Anosov relation on A and A is an isolated subset.*
(2) *The restriction f_A is an Anosov relation on A and A is an expansive subset of X for f.*
(3) *A is an isolated hyperbolic subset of X.*

Proof. (3) $\Rightarrow$ (1) and (2). Let $\gamma > 0$ satisfy (2.4) with $B = A$ and (2.6). Given $\epsilon > 0$, choose $\delta > 0$ so that any δ-chain for f in A can be $\min(\epsilon,\ \gamma)$-shadowed by a 0-chain for f. Thus if x is a δ-chain for f_A, then there exists y$\in X_f$ with

$$d(x_i,\ y_i) \leq \min(\epsilon,\ \gamma) \text{ for all } i \in \mathbb{Z}.$$

By (2.4), it follows that $y_i \in A$ for all $i \in \mathbb{Z}$ and so y$\in A_f$. Thus y is a f_A chain ϵ-shadowing x. This implies that A satisfies the shadowing property for f_A. A is expansive for f_A with the same constant γ. Thus f_A is Anosov. A is isolated and expansive for f by assumption.

(1) and (2) $\Rightarrow$ (3) By Corollary 2.12, A satisfies the shadowing property when f_A is Anosov. By assumption, A is isolated and expansive for f.

(1) $\Rightarrow$ (2) Let $\gamma > 0$ satisfy (2.4) with $B = A$ and (2.6) for f_A. It follows that (2.6) holds for f. That is, if x, y$\in X_f$ and $d(x_i,\ A) \leq \gamma$, $d(y_i,\ A) \leq \gamma$ for all $i \in \mathbb{Z}$, then by (2.4), $x_i,\ y_i \in A$ for all $i \in \mathbb{Z}$. That is, x, y$\in A_f$ and so (2.6) for f_A implies $x_i = y_i$ for all $i \in \mathbb{Z}$.

(2) $\Rightarrow$ (1) Let $\gamma > 0$ satisfy (2.6). Choose $0 < \delta_1 \leq \frac{\gamma}{2}$ so that every δ_1-chain for f_A can be $\frac{\gamma}{2}$-shadowed by some f_A chain. By Lemma 2.11, we can choose $0 < \delta \leq \delta_1$ so that any δ-chain for f in $\overline{V_\delta}(A)$ can be $\frac{\gamma}{2}$-shadowed by a δ_1-chain for f_A. Assume x$\in X_f$ with $d(x_i,\ A) \leq \delta$ for all $i \in \mathbb{Z}$. We prove $x_i \in A$ for all $i \in \mathbb{Z}$ which will imply A is isolated. Since x is a f chain in $\overline{V}_\delta(A)^{\mathbb{Z}}$, it is $\frac{\gamma}{2}$-shadowed by some δ_1-chain y for f_A. Thus y is $\frac{\gamma}{2}$-shadowed by some f_A chain z. In particular, x, y$\in X_f$ with $d(x_i,\ z_i) \leq \gamma$ for all $i \in \mathbb{Z}$ and $z_i \in A$ for all $i \in \mathbb{Z}$. By (2.6) $x_i = z_i$ and so $x_i \in A$ for all $i \in \mathbb{Z}$. $\square$

THEOREM 2.18. *Let f be a closed relation on X with the sample path homeomorphism σ_f on X_f. Let A be a surjective subset of X. Each of* *the following properties holds for A with respect to f if and only if the corresponding property holds for A_f with respect to σ_f.*

(1) *A is isolated.*
(2) *A is expansive.*
(3) *A satisfies the shadowing property.*
(4) *A is hyperbolic.*
(5) *A is Anosov.*

Proof. For (1) we apply Theorem 2.9(b) with $A = B$. For (2) we apply Theorem 2.9(b) to the relation $f \times f$ and the closed subset 1_A and 1_X. Observe that $(1_A)_{f\times f} = 1_{A_f}$. For (3) apply Theorem 2.16. For (4) use (2) and (3). For (5) use (1), (2) and (3), applying Theorem 2.17. □

Now we describe some simple properties.

LEMMA 2.19. *If A is a clopen subset of X, then A is isolated with respect to f. If f is a clopen surjective relation on X, then f satisfies the shadowing property.*

Proof. Since A is a clopen subset of X, there exists a $\gamma > 0$ such that $B(A,\ \gamma) = A$. Let x$\in X_f$ and $d(x_i,\ A) < \gamma$ for all $i \in \mathbb{Z}$. Since $x_i \in B(A,\ \gamma) = A$ for all $i \in \mathbb{Z}$, we have x$\in A_f$. Thus A is isolated with respect to f. Since f is an open subset of $X \times X$, for every $(x,\ y) \in f$ there exists an $\epsilon(x,\ y) > 0$ such that

$$B(x,\ \epsilon(x,\ y)) \times B(y,\ \epsilon(x,\ y)) \subset f.$$

Then $\{B(x,\ \frac{1}{2}\epsilon(x,\ y)) \times B(y,\ \frac{1}{2}\epsilon(x,\ y)) \mid (x,\ y) \in f\}$ is an open cover of f. Since f is compact, there exist finitely many points $(x_1,\ y_1), \cdots,$ $(x_n,\ y_n) \in f$ such that

$$f \subset \cup_{i=1}^n B(x_i,\ \frac{1}{2}\epsilon_i) \times B(y_i,\ \frac{1}{2}\epsilon_i),$$

where $\epsilon_i = \epsilon(x_i,\ y_i)$ for all i. Let $\epsilon = \min\{\frac{1}{2}\epsilon_i \mid i = 1, 2, \cdots, n\}$. To prove that $V_\epsilon \circ f = f$, let $(p,\ q) \in V_\epsilon \circ f$. There exists $r \in X$ such that $(p,\ r) \in f$ and $(r,\ q) \in V_\epsilon$. We can choose i so that $(p,\ r) \in B(x_i,\ \frac{1}{2}\epsilon_i) \times B(y_i,\ \frac{1}{2}\epsilon_i)$. Then $d(p,\ x_i) < \frac{1}{2}\epsilon_i < \epsilon_i$. Since $d(r,\ y_i) < \frac{1}{2}\epsilon_i$ and $d(q,\ r) < \epsilon \le \frac{1}{2}\epsilon_i$, we have

$$d(q,\ y_i) \le d(q,\ r) + d(r,\ y_i) < \frac{1}{2}\epsilon_i + \frac{1}{2}\epsilon_i = \epsilon_i.$$

Thus $(p,\ q) \in B(x_i,\ \epsilon_i) \times B(y_i,\ \epsilon_i) \subset f$ and so $V_\epsilon \circ f \subset f$. Since $f \subset V_\epsilon \circ f$, we have $V_\epsilon \circ f = f$. So any ϵ-chain for f is a 0-chain for f. Hence f has the shadowing property. □

Corollary 2.20. (a) *If X is any compact metric space, then the shift homeomorphism σ on $X^{\mathbb{Z}}$ satisfies the shadowing property.*

(b) *If X is a finite set and f is any relation on X, then σ_f on X_f is an Anosov homeomorphism.*

Proof. (a) Since $f = X \times X$ is a clopen surjective relation on X, by Lemma 2.19, f satisfies the shadowing property. By Theorem 2.18, $\sigma_f = \sigma$ satisfies the shadowing property.

(b) We replace X by $D(f)$ if necessary to assume that f is surjective. Since $X \times X$ is a discrete space, f is a clopen surjective relation on X. By Lemma 2.19 and Theorem 2.18, σ_f satisfies the shadowing property. Since 1_X is a clopen subset of $X \times X$, by Lemma 2.19, 1_X is isolated with respect to $f \times f$ and so f is expansive. By Theorem 2.18, σ_f is expansive. Thus σ_f is an Anosov homeomorphism. □

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